Theory of Distributed Systems

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Exercise 11

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Due: 08.02.2022, **8:15h**

Please submit your solution in PDF format by sending an email to {schmalhofer,varricchio}@em.uni-frankfurt.de. Make sure that your solution reaches us before 8:15 am! Solutions are discussed on Feb 11th, 10:00h - 12:00h (Zoom Meeting-ID: 963 6309 6725, same password as lecture material).
Exercises with * are bonus; they count for your score but not for the sum of points.
This is the last exercise sheet of the lecture.

Exercise 11.1. *c-independence vs. independence number* (6 = 3 + 3 Points)

Show the following for an arbitrary unweighted undirected graph G:

a) If $\alpha(G) \leq c$, then G is c-independent.

b) If G is c-independent, then G has independence number $\alpha(G) = O(c \cdot \log n)$.

Exercise 11.2. *Reducing Energy Costs* (8 = 2 + 2 + 4 Points)

Due to the continuous increase in energy costs, we design a learning-based protocol in wireless networks that penalizes unsuccessful transmissions. To do so, we change the utility $u_i(x^t)$ gained for the chosen action. Now $\beta > 1$ is the new cost for an unsuccessful transmission:

$$u_i(x^t) = \begin{cases} 1 & x_i^t = 1 \text{ and } v_i \text{ successful} \\ -\beta & x_i^t = 1 \text{ and } v_i \text{ not successful} \\ 0 & x_i^t = 0 \end{cases}$$

a) Adapt Lemma 68 to the new scenario, i.e. prove the following lemma:

   Lemma:
   Suppose a history x is such that node v_i has regret $R_i(x) \leq 0$. Then at least a $\frac{\beta}{1+\beta}$-fraction of v_i's transmission attempts have been successful.

Use the lemma from a) to show a result similar to Theorem 38 for the new scenario:

Theorem:
Consider a c-independent conflict graph. Suppose there is a history x such that all nodes v_i have $R_i(x) \leq 0$. Then the average number of successful transmissions is an $O(c \cdot \beta)$-approximation of the optimum.

Let I^* denote a maximum independent set. For $v_i \in I^*$, let $t_i = \sum_{t=1}^{T} x_i^t$ be the number of attempts by node v_i.

b) First, consider the case that at least half the nodes $v_i \in I^*$ have $t_i \geq T/2$. Prove the theorem for this case.

c) Second, take a look at the remaining case that at least half the nodes $v_i \in I^*$ have $t_i < T/2$. Prove the theorem for this case.

Exercise 11.3. *Random Walks* (6 = 2 + 3 + 1 Points)

We show that the bound of Theorem 40 is asymptotically tight. Let us analyze the Random-Walk Protocol on a complete graph G with n vertices. One node in the complete graph is a source node s and one is a target node $t \neq s$. Initially, all $\ell_s^0 = m$ load units are located at s, and $\ell_v^0 = 0$ for all $v \neq s$. Node t has $\tau_t = m$ free slots, and every other node has no free slot, $\tau_v = 0$ for $v \neq t$.

a) Show that the hitting time is $H(G, s, t) = n - 1$. Make use of symmetry.

Let X_i be the number of rounds until load unit i reaches t, for $i = 1, \ldots, m$.

b) Show that there is a constant $c > 0$ such that

$$\Pr\left[\max_{i=1,\ldots,m} X_i > (n-1)\log_4 m\right] \geq c.$$

You may assume that $n \geq 3$.

Hint: The inequality $\left(1 - \frac{1}{n-1}\right)^{n-1} \geq 1/4$ for each $n \geq 3$ can be helpful.

c) Conclude from a) and b) that the expected number of rounds until a balanced distribution is reached is at least $c \cdot H(G) \cdot \log_4 m$.

Exercise 11.4. *Lower Bounds for Push – Bonus* (6* = 3 + 3 Points)

Show the following statements for the Push protocol:

a) Whp it requires $\Omega(\log n)$ rounds to inform all nodes.

b) For the n-node star graph, the expected number of rounds to inform all nodes is $\Omega(n \log n)$.

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