Theory of Distributed Systems

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Exercise 10

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Due: 01.02.2022, **8:15h**

Please submit your solution in PDF format by sending an email to {schmalhofer,varricchio}@em.uni-frankfurt.de. Make sure that your solution reaches us before 8:15 am! Solutions are discussed on Feb 4th, 10:00h - 12:00h (Zoom Meeting-ID: 963 6309 6725, same password as lecture material).
Exercises with * are bonus; they count for your score but not for the sum of points.

Exercise 10.1. *Max-Average-Degree and LogColor* (3 and 4* Points)

a) Give an example of a graph G with n nodes with $MaxAvg(G) = \Theta(n)$ and $\chi(G) = O(1)$. Here, $\chi(G)$ is the chromatic number of the graph.

b*) Show that there is a weighted n-node graph where LogColor needs $\Omega(n \log n)$ rounds with high probability.

Exercise 10.2. *Non-constant* $\rho(G)$ (4 Points)

Give an example of a graph G with n nodes having $\rho(G) \geq \sqrt{n}$, i.e., in *every* ordering π of the nodes, there is at least one node v with $\Omega(\sqrt{n})$ independent neighbors that are placed before v in π.

Exercise 10.3. *Line- and Square-Graphs* (8 = 2 + 3 + 3 Points)

Suppose there are n base stations located along a line. Each base station tries to reach mobile receivers in the vicinity on the line.

Formally, we assume for each base station i there is a continuous line segment of length $\ell_i > 0$ and the base station is located in the middle of this segment. Two base stations are conflict-free if and only if their segments do not intersect, i.e. no point belongs to both segments.

The resulting conflict graph G can be seen as a "one-dimensional disk-graph". Show the following bounds on the inductive independence number $\rho(G)$:

a) $\rho(G) \leq 2$

b) $\rho(G) \leq 1$

Now we generalize the previous model to two dimensions: Assume every base station i is a center of a square with side length $\ell_i > 0$. All squares have the same orientation. Show the following bound for the new conflict graphs G'.

c) $\rho(G') \leq 4$

Hint: Obviously, a solution to b) is sufficient to solve a) as well.

Exercise 10.4. *No-Regret Algorithms* (4 Points)

In this exercise we consider a slightly more general model of online learning: Instead of having two actions, there is a set $\{1, \ldots, K\}$ of K actions. In each round $t = 1 \ldots T$, the algorithm has to select one of these actions. After selection, the algorithm sees the utility $u_k^t \in [0, 1]$ of every action $k \in \{1, \ldots, K\}$ in that round.

Consider the following algorithm A:

In every round $t = 1 \ldots T$, the smallest action $k \in \{1, \ldots, K\}$ with maximum cumulated utility $U_k^{t-1} = \sum_{j=1}^{t-1} u_k^j$ so far is selected. Note that in the first round action 1 is always selected.

The regret of A is

$$R_A(T) \;=\; \max_{k \in \{1,\ldots,K\}} \sum_{t=1}^{T} u_k^t \quad - \quad \sum_{t=1}^{T} u_{A(t)}^t \;,$$

where $A(t)$ is the action selected by A in round t.

A fulfills the no-regret property if for $T \to \infty$, the average regret $R_A(T)/T \to 0$ for every realization of utilities.

Prove or disprove: A is a no-regret algorithm.

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