Theory of Distributed Systems

Winter Term 2021/22

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# Exercise 8

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Due: 18.01.2022, **8:15h**

*Please submit your solution in PDF format by sending an email to {schmalhofer,varricchio}@em.uni-frankfurt.de. Make sure that your solution reaches us before 8:15 am! Solutions are discussed on Jan 21st, 10:00h - 12:00h (Zoom Meeting-ID: 963 6309 6725, same password as lecture material).*
**Exercises with * are bonus; they count for your score but not for the sum of points.**

**Exercise 8.1.** *Valiant's Trick on the Cube* (9 = 2 + 2 + 3 + 2 and 2* Points)

Assume Valiant's trick is applied to the cube $M(\ell, 3)$, i.e., every node selects an (intermediate) target node independently uniformly at random. Routes are selected dimension-by-dimension. To simplify the analysis, each undirected edge is replaced by two directed edges. A directed edge is called $(i, j)$-*increasing* if it increases dimension $i$ from $j$ to $j + 1$. Similarly, an edge decreasing dimension $i$ from $j$ to $j - 1$ is called $(i, j)$-*decreasing*. For example, the edge $(7, 3, 0) \to (7, 4, 0)$ is $(1, 3)$-increasing, while the edge $(7, 3, 0) \to (7, 2, 0)$ is $(1, 3)$-decreasing.

a) Give the expected congestion $\mathbb{E}[C(e)]$ of any $(i, j)$-increasing edge $e$ in terms of $i, j$ and $\ell$.

b) Show that $\mathbb{E}[C(e)] \leq \ell/4$ for every directed edge $e \in E$.

c) Show that the congestion is $O(\ell)$ with probability at least $1 - 2^{-\Omega(\ell)}$.

d) Show that the congestion is $\Omega(\ell)$ with probability at least $1 - 2^{-\Omega(\ell)}$.

e*) Show the following stronger result: The congestion is $\Omega(\ell)$ with probability at least $1 - 2^{-\Omega(\ell^2)}$.

*Hint*: In every exercise you can assume that $\ell$ is even!

**Exercise 8.2.** *h-relation* (5 Points)

Prove Lemma 51 of the lecture notes:

Using Valiant's Trick for routing an arbitrary $h$-relation on the hypercube, the congestion is $C = \mathcal{O}(\log n + h)$ whp.

**Exercise 8.3.** *GrowingRank* (5 Points)

Solve the exercise in the proof of Lemma 55 in the lecture notes:

Show that packets $p_0, \ldots, p_s$ are distinct, i.e., no packet appears more than once in the delay sequence.

**We wish everyone happy holidays and great start into 2022!**