Theory of Distributed Systems

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Exercise 3

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Due: 16.11.2021, **8:15h**

Please submit your solution in PDF format by sending an email to {schmalhofer,varricchio}@em.uni-frankfurt.de. Make sure that your solution reaches us before 8:15 am! Solutions are discussed on Nov 19th, 10:00h - 12:00h (Zoom Meeting-ID: 963 6309 6725, same password as lecture material).
Exercises with * are bonus; they count for your score but not for the sum of points.

Exercise 3.1. *Anonymous Paths* (7 = 3 + 4 and 3* Points)

Consider the synchronous CONGEST model and an anonymous path (without IDs, nodes only know their degree) with simultaneous wakeup. We say the path is *non-oriented* if nodes cannot distinguish correctly between left and right neighbor. Similarly, we say the path is *oriented* if every node can distinguish correctly between left and right neighbor.

a) Show that there is a deterministic algorithm that 2-colors every anonymous oriented path.

b) Show that there is no deterministic algorithm that 2-colors every anonymous non-oriented path.

c*) Describe how to solve the problem in b) when every vertex is able to flip fair coins (every vertex can generate a random bit that equals 0 or 1 with same probability).

Exercise 3.2. *Optimization and Distributed Algorithms* (3 and 5* Points)

For each of the following two exercises you should construct a class of graphs $G = (V, E)$ with suitable ID assignment such that the described property holds. We use n to denote the number of vertices.

a) The size of a *maximum independent* set in G is $\Omega(n)$ times the one computed by MIS-Rank.

b*) **Reduce** needs a number of colors that is $\Omega(n)$ times the *chromatic number* $\chi(G)$.

Exercise 3.3. *Tree-coloring in $O(1)$, and Linial's Lower Bound* (5 = 2 + 3 Points)

Consider the synchronous LOCAL model.

a) Show that there is a $\mathcal{O}(\log n)$-coloring algorithm which needs $\mathcal{O}(1)$ rounds on trees where every node knows its parent.

b) Disprove the following statement:

If there exists a k-ary q-coloring function B, then there is a $(k-1)$-ary $2q$-coloring function B'.

Hint: What would be the consequence of this statement in the proof of Linial's lower bound?

Exercise 3.4. *Coloring* (6 = 3 + 3 Points)

Consider the generalization of the 6-Color algorithm from the lecture to arbitrary, bounded degree graphs. The algorithm concatenates for all $w \in \Gamma(v), w \neq v$ the strings which the 6-Color algorithm would have chosen as new color of node v if neighbor w was v's parent in the tree. This reduction step, starting with a legal K-bit coloring, yields a $\Delta \cdot (\lceil \log K \rceil + 1)$-bit coloring. Assume that the resulting coloring has exactly this length, i.e., each of the entrys that could possibly be shorter is filled with leading zeros.

a) Prove that the coloring obtained from this reduction step is legal.

b) Show that the minimum number of bits that can be achieved by iterating this reduction step is of order $\Theta(\Delta \log \Delta)$, i.e., prove that there are constants $c > c' > 0$ such that for any length more than $c \cdot \Delta \log \Delta$ the process reduces the length and for a length $c' \cdot \Delta \log \Delta$ the process does not reduce the length.

This result is given as Theorem 7.3.8 in Peleg's book. Note that Peleg's estimate of 2Δ instead of $O(\Delta \log \Delta)$ is a bit too optimistic.

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