Optimization and Uncertainty

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Assignment 8

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Exercise 8.1 *Online* PERSUADE (4 + 2 + 4 = 10 points)

Consider n boxes and assume that all boxes and distributions $\mathcal{D}_i$, $i = 1, ..., n$, are known. The boxes are opened in a known order. In round i, $\mathcal{S}$ opens box i and sends a signal to $\mathcal{R}$ ("take box i" or "do not take box i"). If $\mathcal{R}$ does not take the box, the process continues with the next box. Otherwise, if $\mathcal{R}$ takes the box, the process stops. Note that when sending the signal in round i, $\mathcal{S}$ only knows the content of boxes $1, ..., i$.

a) For IID boxes, show that there exists a direct and persuasive $(1-1/e)^{-1}$-competitive algorithm. *Hint: Use a modification of Algorithm 12 from the lecture notes where, instead of a random box, the modified algorithm takes either the first yes-box or the last no-box (the rest remains unchanged).*

b) For independent (but not necessarily identical) boxes and the SSQ condition fulfilled with the SSQ box in round $i^* = n$, show that there exists an algorithm with constant competitive ratio.

c) For independent (but not necessarily identical) boxes and the SSQ condition fulfilled with the SSQ box in round $i^* < n$, show that there is no algorithm with finite competitive ratio.

Exercise 8.2 DELEGATION (2 + 1 + 1 = 4 points)

A class of instances for DELEGATION with $n \geq 2$ IID boxes is given as follows:

For each box $i \in [n]$, a random prize-pair $(s_i, r_i) \in \{0,1\}^2$ is obtained with two independent Bernoulli trials such that $\Pr[s_i = 1] = \Pr[r_i = 1] = \frac{1}{n}$ and $\Pr[s_i = 0] = \Pr[r_i = 0] = 1 - \frac{1}{n}$.

Assume that $\mathcal{S}$ breaks ties in favor of $\mathcal{R}$.

a) Apply Algorithm 14 from the lecture on this class of instances. Show your calculations for each step of the algorithm and write down the distribution of the computed decision scheme ψ.

b) Calculate the expected reward for $\mathcal{R}$ when using Algorithm 14 for computing a decision scheme.

c) Find a best possible decision scheme for $\mathcal{R}$ for this class of instances or argue why Algorithm 14 yields a best possible decision scheme here.

Exercise 8.3 *Online* DELEGATION (1 + 2 + 3 = 6 points)

Consider an online variant of DELEGATION where n independent boxes arrive sequentially in a fixed order. Both the sender $\mathcal{S}$ and the receiver $\mathcal{R}$ know the order of the boxes and their respective distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in advance, and $\mathcal{R}$ commits to a decision scheme ψ. At arrival of box i, $\mathcal{S}$ looks in the box (i.e., sees prize-pair $\theta_{ij} = (s_{ij}, r_{ij})$ randomly drawn according to distribution $\mathcal{D}_i$) and decides immediately whether to recommend it to $\mathcal{R}$ or not. If $\mathcal{S}$ lets it pass (and $i < n$), the next box $i + 1$ arrives. The process ends when $\mathcal{S}$ recommends a box (upon which $\mathcal{R}$ makes the accept/reject decision according to ψ) or if $\mathcal{S}$ has let all n boxes pass.

The approximation factor of a decision scheme for this problem is defined with respect to the expected optimal prize that $\mathcal{R}$ could get in an online scenario without delegation on the same instance.

a) Consider the following decision scheme ψ: Choose a single box $i^* := \arg\max_{i\in[n]} \mathbb{E}[r_{ij}]$ from which the maximum expected prize for $\mathcal{R}$ is obtained. Always accept any recommendation from that box i^* and reject everything else, i.e., $\psi(\theta_{i^*j}) = 1$ for all $j \in m_{i^*}$ and $\psi(\theta_{ij}) = 0$ for all $i \neq i^*, j \in m_i$.

   Show that this decision scheme is always n-approximative.

b) Consider an instance of Online DELEGATION with some box $i < n$ and prize-pair θ_{ij}. Moreover, there is another box $i' > i$ with a prize-pair $\theta_{i'j'}$ such that $s_{i',j'} \geq n^2 \cdot s_{ij}$ and $\Pr[\theta_{i'j'}] \geq \frac{1}{n}$.

   Prove: For any decision scheme ψ with $\psi(\theta_{i'j'}) = 1$, it holds that $\mathcal{S}$ never recommends θ_{ij}.

c) Construct a class of instances of Online DELEGATION where every decision scheme is $\Omega(n)$-approximative. Show the correctness of your construction.

   Hint: Use insights from b).