Optimization and Uncertainty

Summer term 2023

Prof. Dr. Martin Hoefer
Conrad Schecker, Lisa Wilhelmi

GOETHE UNIVERSITÄT FRANKFURT AM MAIN
Institute of Computer Science
Algorithms and Complexity

Assignment 5

Issued: 23.05.2023
Due: 30.05.2023, **10:00h**

Exercise 5.1 *Prophet Independent Set* (2 + 5 + 2 = 9 points)

a) Show Proposition 1 from the lecture notes:
Given any graph G and an ordering π of the nodes, consider the standard greedy algorithm that considers all nodes in the order of π and adds them to the indepedent set whenever possible. The algorithm computes a ρ_π-approximation.

b) Consider general disk graphs $G = (V, E)$, i.e., each node $v_i \in V$ is the center of a disk of radius $r_i \in \mathbb{R}$. Adjust Algorithm 6 from the lecture in the following way: The Independent Set M_1 is computed applying the algorithm in (a). Furthermore, add every element in M_2 to M_3 with probability $1/(2 \cdot \rho_\pi)$. Assume that the remaining steps of the algorithm are unchanged. Show that this adjustment yields a $4\rho_\pi^2$-approximation in disk graphs.
Hint: You can gear your analysis towards the proof of Theorem 16 in the lecture notes. Which of the lemmas 5-7 need to be adjusted and how?

c) Using the result in (b), show that there exists a 100-competitive algorithm for online IndependentSet in disk graphs.

Exercise 5.2 *Proof for Secretary Scheduling* (4 points)

Complete the proof of Lemma 8 in the lecture notes: Show that the simulation yields the same distribution as the original process in the Secretary case. Namely, proof that

(1) Given any sample length $k \in \{0, 1, ..., n\}$, the arrival order of V in the simulation is uniform over all permutations.

(2) Given any arrival order, the sample length in the simulation is binomially distributed with parameters n and $p = 1/2$.
Hint: Use Bayes' theorem.

Exercise 5.3 *Secretary Weighted Interval Scheduling II* (4 points)

Consider another extension of the Secretary Interval Scheduling problem. Here, every node $u \in V$ is assigned a value $v(u) \geq 0$ that is **not** known in advance, but revealed upon arrival of node u (together with all edges to nodes that arrived in earlier rounds). The goal is to compute an independent set with maximum total value.

Algorithm 7 discussed in the lecture can be adjusted to this problem in the following way: Fix the length of the sample phase $k = n/2$. Let M_1 be computed by an optimal dynamical program (DP). Thus, u_t is added to M_2 in the first filtering step if DP on $V^S \cup \{u_t\}$ would have selected u_t. Assume that the remaining steps of the algorithm are unchanged. Show that the competitive ratio of this algorithm is $\Omega(n/\log n)$.

Exercise 5.4 *k-ProbeMax with unprobed boxes* (3 points)

Consider an extension of the k-ProbeMax problem with n boxes. After $k < n$ boxes have been opened, either one of the probed boxes or an unprobed box can be chosen in the end. As in the original problem, the reward for a probed box is given by its observed realization. Whenever an unprobed box is chosen, the expected value of the box is obtained. Show that the adaptivity gap for this variant is constant.
Hint: Compare an optimal policy for this variant to an optimal policy for the original k-ProbeMax problem.

Contacts: {schecker,wilhelmi}@em.uni-frankfurt.de.