Optimization and Uncertainty

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Assignment 4

Issued: 09.05.2023
Due: 23.05.2023, **10:00h**

This assignment is due only in **two weeks**.

Exercise 4.1 *Markov Decision Processes II* (4 points)

Consider a generalization of the version of Markov decision processes covered in the lecture. Assume that for every state $s \in S$, only a subset of the actions $\mathcal{A}_s \subseteq \mathcal{A}$ with $\mathcal{A}_s \neq \emptyset$ is available.

Devise an algorithm that computes an optimal policy for a finite time horizon T. Show the correctness of your algorithm and give a bound on its running time.

Hint: A formal proof of the correctness of your algorithm is required.

Exercise 4.2 *IndependentSet in Unit Disk Graphs* ($4 + 2 + 2 = 8$ points)

A unit disk graph is an undirected graph $G = (V, E)$ with $V \subseteq \mathbb{R}^2$ being a collection of points in the Euclidean plane. Each $v \in V$ is the center of a closed unit disk, i.e., the set of all points whose Euclidean distance to v is at most 1. There is an edge between two vertices $v_i, v_j \in V$ with $v_i \neq v_j$ if and only if their closed unit disks intersect.

a) For each $v \in V$, let $N(v)$ be the set of neighbors of v in G. Show that there are at most five vertices in $N(v)$ with mutually non-intersecting closed unit disks.

b) Show that there exists a 5-competitive deterministic algorithm for online IndependentSet in unit disk graphs.

c) Show that the competitive ratio of any deterministic algorithm for online IndependentSet in unit disk graphs is at least 5, even if the number of nodes is known in advance.

   Hint: Use sketches to visualize your arguments.

Exercise 4.3 *Online* Interval Scheduling (1 + 5 = 6 points)

Let $\mathcal{I}$ be a set of n (unknown) requests for tasks being processed on a single machine. Suppose these requests arrive in an online manner: At each time step, a request comes in and reveals its individual start and completion time. The request must be either accepted or declined. If it is accepted, then it occupies the machine for the specified period of time. A request cannot be accepted if its period intersects the period of a previously accepted request. The goal is to accept as many requests as possible.

a) For instances without constraints on the length of the intervals, show that the competitive ratio of any deterministic algorithm is $\Omega(n)$.

b) Show that any randomized online algorithm has competitive ratio $\Omega(n)$ as well, even if the number n of intervals is known in advance.
Hint: Use Yao's principle.

Exercise 4.4 *Secretary Weighted Interval Scheduling I* (1 + 2 + 4 = 7 points)

Consider a variant of the Secretary Interval Scheduling problem where each interval $v \in V$ has some non-negative weight $w_v \geq 0$. The weights of all $n = |V|$ intervals are known in advance, but the actual intervals are revealed in an online fashion in uniform random order.

a) Let $w_{\max}$ be the maximum weight of all intervals in V, i.e., $w_{\max} = \max_{v \in V} w_v$. Furthermore, let $V_0 := \left\{ v \in V : w_v \leq \frac{w_{\max}}{4n} \right\}$.

   Prove that any maximum-weight independent set of the subgraph induced by V_0 has a value of at most $\frac{w_{\max}}{4}$.

b) Let $V' \subseteq V$ be any subset of V, and let $w'_{\min}$ and $w'_{\max}$ denote the minimum and maximum weight of the intervals in V', respectively.

   Prove: If $w'_{\max} \leq c \cdot w'_{\min}$ for some constant $c \geq 1$, then Algorithm 7 is $8c$-competitive for the weighted instance induced by V'.

c) Design an algorithm for the described weighted variant of the problem that is $\mathcal{O}(\log n)$-competitive. Prove your answer.

The assignments and further information on the course are provided on our website:
https://algo.cs.uni-frankfurt.de/lehre/oau/sommer23/oau23.shtml

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