Optimization and Uncertainty

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Assignment 2

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Due: 02.05.2023, **10:00h**

Exercise 2.1 *Chernoff Inequality* (2 + 3 = 5 points)

a) An unfair coin turns up heads with probability $\frac{4}{5}$ and tails with probability $\frac{1}{5}$. Using the Chernoff inequality, estimate the upward probability that tails turns up at least five times in a sequence of ten coin flips.

Hint: Use the definition of the inequality in the appendix of the lecture notes:

$$\Pr[X \geq (1+\delta) \cdot \mathbb{E}[X]] \leq e^{-\mathbb{E}[X] \cdot \delta^2/3} .$$

b) Consider an unfair coin with probabilities $\frac{4}{5}$ and $\frac{1}{5}$. Unfortunately, you forgot whether heads or tails is more likely. Depict a method that assings the probabilities to the events with n coin tosses and an error probability of at most $e^{-\mathcal{O}(n)}$.

Exercise 2.2 *NP-Hardness of offline* Item Allocation (4 points)

Show that the offline version of the Item Allocation problem is NP-hard.
Hint: Use IndependentSet *for the reduction.*

Exercise 2.3 *Greedy Matching* (5 points)

Consider a graph $G = (V, E)$. Every edge $e \in E$ has a value $v_e \geq 0$. A *matching* is a set of edges $M \subseteq E$ where any node $u \in V$ is incident to not more than one edge in M, i.e., $|\{e \in M, u \in e\}| \leq 1$. The value of M is defined by $v(M) = \sum_{e \in M} v_e$.
Let M^* be a matching with maximum value. The *GreedyAlgo* starts with $M_g = \emptyset$ and iterates over all edges consecutively in non-ascending order of their values. An edge is added to M_g if M_g is still a matching thereafter. Show that the resulting greedy matching M_g satisfies

$$v(M_g) \geq \frac{1}{2} \cdot v(M^*) .$$

Exercise 2.4 *Secretary Matching with approximation* (3 + 3 = 6 points)

Let *GreedyAlgo* be defined as in the previous exercise. Recall that it computes a 2-approximation for the offline weighted Matching problem.

Consider the following variation of the algorithm for Secretary Matching (Algorithm 2) discussed in the lecture: In line 8, *GreedyAlgo* is applied to compute a matching $M^{g,t}$ of $G_t = (L_t \cup R, E_t)$ in each round $t \geq s + 1$. Consistently $M^{g,t}$ is used in the following lines (instead of $M^{*,t}$).

a) Show that Lemma 1 takes the following form given the modification described above: *For every given round* $t = s + 1, \ldots, n$*, we have* $\mathbb{E}[v(e_t)] \geq v(M^*)/(2 \cdot n)$.

   Hint: Go over the proof of Lemma 1 and explain for every step which adjustments are necessary.

b) Show that the described variation of the algorithm is $2 \cdot (\mathsf{e} + o(1))$-competitive.

   Hint: Adapt the proof of Theorem 8. Use the result from a). What about Lemma 2?

Commentary: This proof can be extended to arbitrary deterministic α-approximation algorithms for offline weighted Matching, where $\alpha > 1$. In this way, one obtains an $\alpha \cdot (\mathsf{e} + o(1))$-competitive algorithm for Secretary Matching.