Optimization and Uncertainty

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Assignment 6

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Exercise 6.1 *Stochastic* Knapsack (4 + 4 points)

In the Stochastic Knapsack problem, like in the non-stochastic variant, items of values v_i and sizes s_i are packed into a knapsack of a fixed capacity c; the goal is to maximize the sum of values of packed items. In the stochastic variant, the sizes are random. The size of an item is revealed only after it has been packed into the knapsack. If it exceeds the knapsack capacity, the item is removed from the knapsack and no further items can be packed. Again, optimal policies are adaptive, meaning that their choices depend on the sizes of already packed items.

a) Show that every policy corresponds to a solution of the following linear program:

$$\begin{aligned}\text{maximize} \sum_i v_i x_i \\ \text{subject to} \sum_i x_i \cdot \mathbb{E}[\min\{s_i, c\}] \leq 2c \\ 0 \leq x_i \leq 1 \text{ for all } \ i.\end{aligned}$$

Hint: Construct a sequence of random variables $Y_{t\in\mathbb{N}}$ such that $\mathbb{E}[Y_{t+1} \mid Y_t] = Y_t$. Assume that, in this case, $\mathbb{E}[Y_t] = \mathbb{E}[Y_0] = 0$ for any $t \geq 0$.

b) For simplicity, assume that $s_i \leq c/2$ for all i with probability 1. Now, use an optimal LP solution x^* to determine a randomized non-adaptive policy as follows: Pack item i into the knapsack with probability $x_i^*/8$. Show that this policy achieves an expected reward of at least $\frac{1}{16}\sum_i v_i x_i^*$.

Exercise 6.2 *Adaptivity gap for k-*TestMax (4 points)

Design a non-adaptive algorithm for k-TestMax in the IID scenario and show that the adaptivity gap is $\mathcal{O}(\log \min(n, k))$.
Hint: First, derive the probability of finding a good item and choose the number of tests per item accordingly next.

Exercise 6.3 PANDORA BOX *Matching* (4 points)

In order to generalize the Pandora Box setup from the lecture, suppose the task is to match people $i \in [n]$ to boxes $j \in [m]$, where each person can take at most one prize home. Person i's value v_{ij} for the prize in box j is independently drawn from a distribution D_{ij}, but it costs c_{ij} to inspect the exact value of the box v_{ij}. Consider $A_{ij}, I_{ij}, \omega_{ij}, \kappa_{ij}$, and b_{ij} to be the corresponding generalizations of the variables introduced in the lecture. Show that for any policy π, the expected reward R is given by

$$R(\pi) = \sum_{i \in [n], j \in [m]} \mathbb{E}\left[A_{ij} \cdot \kappa_{kj} - (I_{ij} - A_{ij}) \cdot b_{ij}\right].$$

Exercise 6.4 *Fair cap* (2 points)

Consider the following distribution for the prize of box i: the prize v_i is equal to w_i with probability q_i and is 0 else. Compute the fair cap.