Optimization and Uncertainty
Summer term 2021

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Assignment 4

Issued: 11.05.2021
Due: 25.05.2021, **10:00h**

The handling time of this assignment is **two weeks**. It will be discussed in the exercise session on 28.05.2021. There will be **no exercise session on 21.05.2021**.

Exercise 4.1 *Online* INTERVAL SCHEDULING (1 + 5 points)

Let $\mathcal{I}$ be a set of n (unknown) requests for task being processed on a single machine. Suppose these requests arrive in an online manner: At each time step, a request comes in and reveals its individual start and completion time. The request must be either accepted or declined. If it is accepted, then it occupies the machine for the specified period of time. A request cannot be accepted if its period intersects the period of a previously accepted request. The goal is to accept as many requests as possible.

a) For instances without constraints on the length of the intervals, show that the competitive ratio of any deterministic algorithm is $\Omega(n)$. An informal explanation is sufficient.

b) Show that any randomized online algorithm has competitive ratio $\Omega(n)$ as well, even if the number n of intervals is known in advance.
Hint: Use Yao's principle.

Exercise 4.2 INDEPENDENT SET *in unit disk graphs* (4 + 3 points)

A unit disk graph $G = (V, E)$ is formed from a collection of points $v_i \in \mathbb{R}^2$ in the Euclidean plane where each v_i is the center of a disk of radius $r_i = 1$. There is an edge between two points v_i, v_j if their disks intersect (which also includes the case of overlapping boundaries).

a) Let C be a disk and $\mathcal{M}_C$ the set of all disks that intersect C. Show that there are not more than five disks in $\mathcal{M}_C$ that are mutually non-intersecting.

b) For online INDEPENDENT SET in unit disk graphs, show that there exists a 5-competitive deterministic algorithm.

Exercise 4.3 *Markov Decision Processes I* (1 + 3 points)

Consider a stochastic decision problem similar to the one with the envelopes solved in the lecture. There are n boxes where box i contains a prize of 1 Euro with probability q_i and is empty otherwise. The game ends when a non-empty box is opened. At each point in time, one can also decide to stop playing. This means, the final prize is either 0 Euros or 1 Euro. Opening box i costs c_i Euros. The goal is to maximize the final prize minus the costs for opening.

a) Model this problem as a Markov decision process. In particular, give the state and action sets as well as transition probabilities and rewards.

b) Derive an optimal policy for this problem.

Exercise 4.4 *Markov Decision Processes II* (3 points)

Consider a generalization of the version of Markov decision processes covered in the lecture. Assume that, for every state $s \in S$, only a subset of the actions $\mathcal{A}_s \subseteq \mathcal{A}$, where $\mathcal{A}_s \neq \emptyset$, is available. Devise an algorithm that computes an optimal policy for a finite time horizon T, show its correctness, and give a bound on its running time.

Exercise 4.5 *Two-round IID Prophets* (3 + 4 points)

Consider two rounds of an IID PROPHET problem. The finite distribution $\mathcal{D}$ has support $\{0, 1, n\}$, $n > 1$. The probabilities are $\Pr[v_1 = 1] = q$, $\Pr[v_1 = n] = (1-q)/n$, and $\Pr[v_1 = 0] = (1-1/n)(1-q)$. The same holds for v_2.

a) Let ALG^* be the optimal online algorithm for this instance. Determine $\mathbb{E}[v(ALG^*)]$ as a function of q and n.

b) Derive a lower bound on the competitive ratio as a function of q and n. What are (asymptotically) optimal choices for q and n when we want to maximize this lower bound? What is the resulting bound? Prove your answer.

The assignments and further information on the course are provided on our website:
`http://algo.cs.uni-frankfurt.de/lehre/oau/sommer2021/oau21.shtml`

Contacts for submissions and questions: {`koglin,wilhelmi`}`@em.uni-frankfurt.de`.