Approximation Algorithms

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Assignment 3

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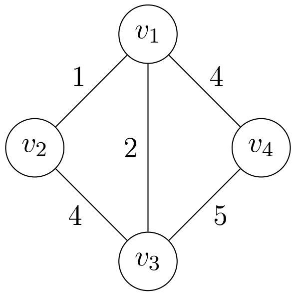
Due: 16.11.2021, **10:15h**

Exercise 3.1 *Metrics* (2 + 2 points)

a) Consider the edge-weighted graph below. Let the *distance* $d(u, v)$ of two vertices u, v be the smallest sum of edge weights of any u-v-path in the graph. Determine whether or not the distance function d forms a metric for the given graph.



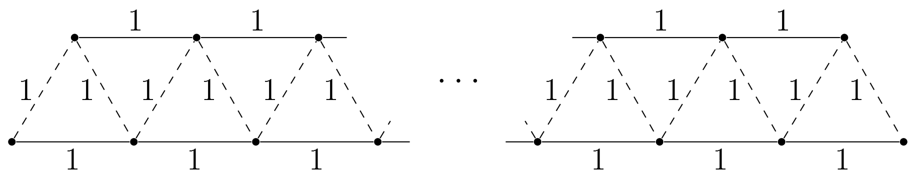
b) Consider the following two classes of distance functions d with the ground set $X = \{1, 2, \ldots, n\}$:

i) $\forall i \neq j : \; d(i, j) = 1$
ii) $\forall i \neq j : \; 0 \leqslant d(i, j) \leqslant 1$

Throughout, assume that $\forall\, i \in X : d(i, i) = 0$ and $\forall\, i, j$ with $i \neq j : d(i, j) = d(j, i)$. Determine which classes of d form a metric.

Exercise 3.2 *Christofides-Serdyukov Algorithm* (3 points)

Consider the following graph with n vertices (where the edges represent Euclidean distances) as input to the *Christofides-Serdyukov* algorithm. Assume that *Kruskal's* algorithm selects the dashed edges before the solid edges. Derive a lower bound on the approximation ratio of the *Christofides-Serdyukov* algorithm.

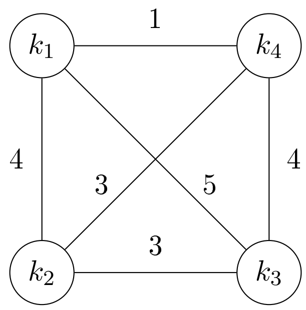


Exercise 3.3 *Nearest-Insertion* (2 + 3 + 2 + 2 + 1 + 2 + 1 points)

For metric MinTSP (Δ-MinTSP), an undirected, edge-weighted, complete graph $G = (V, E)$ is given, where $|V| = n$. For any pair of vertices $u, w \in V$, the edge weight is regarded as their metric distance $d(u, w) \geq 0$, where $d(u, u) = 0$. Let $u \to w$ denote a movement from u to w along $\{u, w\}$. A tour S on G is a set of consecutive movements such that starting in a vertex v eventually leads back to v, where any vertex visited by S in between is visited only once. With a slight abuse of notation, $d(S)$ denotes the total length of S, i.e., the sum of all covered distances.

Consider the *Nearest-Insertion* algorithm mentioned in the lecture, which in detail operates as follows: In the first step, it determines the pair of vertices $u, w \in V$ with minimum distance and constructs the tour $S_1 = \{v_1 \to v_1\}$, where v_1 is selected uniformly at random between u and w. In step i, $2 \leq i \leq n$, a new vertex $v_i \in V$ with minimum distance $d(v_i, v)$ to a node v visited by S_{i-1} is chosen. Define $c(S_{i-1}, v_i) = \min_{(u \to w) \in S_{i-1}} d(u, v_i) + d(v_i, w) - d(u, w)$ as the minimum elongation of S_{i-1} by adding v_i to the tour. If the minimum is achieved for $v \to v' \in S_{i-1}$, *Nearest-Insertion* sets $S_i = (S_{i-1} \setminus \{v \to v'\}) \cup \{v \to v_i, v_i \to v'\}$.

a) Perform *Nearest-Insertion* on the following input graph, where $V = \{k_1, k_2, k_3, k_4\}$ and the numbers next to the edges denote the shortest distances between the respective vertices. Assume there exists an order $k_1 < k_2 < k_3 < k_4$ over V. Here, choose v_1 in the first step such that it is the lower vertex of the two vertices with minimum distance. Furthermore, break ties between multiple possible paths $u \to w \in S_{i-1}$ minimizing $c(S_{i-1}, v_i)$ in favor of those with lowest u according to this order. For each step $1 \leq i \leq 4$, state v_i, S_i, and $d(S_i)$.



In the remainder of this exercise, the goal is to show that the approximation ratio of *Nearest-Insertion* for Δ-MinTSP on general input graphs is $2 \cdot \left(1 - \frac{1}{n}\right)$. To do so, take the following course of actions.

(b) Show that for any tour S visiting less than n distinct vertices, any vertex v visited by S, and any vertex v' not visited by S, it holds that $c(S, v') \leq 2 \cdot d(v, v')$.
Hint: Notice that $c(S, v')$ is the minimum elongation. Use the triangle inequality.

(c) Show that for any step $2 \leq i \leq n$, for any vertex v visited by S_{i-1}, and any vertex v' not visited by S_{i-1}, it holds that $c(S_{i-1}, v_i) \leq 2 \cdot d(v, v')$.

(d) Show that $d(S_i)$ can be written as the sum Λ_i of all elongations that have occurred by the end of step i, for any step $2 \leq i \leq n$.

(e) Suppose that a minimum spanning tree M for G is given. Let $d(M)$ denote the sum of the lengths of all edges in M. Assume that, for all $2 \leq i \leq n$, the vertex v_i chosen by *Nearest-Insertion* in step i can be assigned one-to-one to an edge e_i in M such that S_{i-1} visits exactly one of the two vertices incident to e_i. Show that $d(S_n) \leq 2 \cdot d(M)$.

(f) Let S^* be an optimal tour visiting all of the n vertices. Show that $d(M) \leq \left(1 - \frac{1}{n}\right) \cdot d(S^*)$.

(g) Show that the approximation ratio of *Nearest-Insertion* is $2 \cdot \left(1 - \frac{1}{n}\right)$.

Note: For any of the tasks, you may assume that the previous statements are true even if you did not show them.

The assignments and further information on the course are provided on our website:
https://algo.cs.uni-frankfurt.de/lehre/apx/winter2122/apx2122.shtml

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