# Algorithmic Game Theory

Winter Term 2023/2024

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# Assignment 4

## Exercise 4.1.

A state of a game is called *strong* Nash equilibrium if there is no coalition of players that can improve by simultaneously deviating to another strategy. Formally, a state S of a game is a strong Nash equilibrium if there is no set of players  $B \subseteq \mathcal{N}$  such that the players in B have a strategy profile  $S'_B = (S'_i)_{i \in B}$  that satisfies  $c_i(S'_B, S_{-B}) \leq c_i(S)$  for all  $i \in B$  and at least one inequality is strict.

Show that there is a strong Nash equilibrium in every correlated matching game.

## Exercise 4.2.

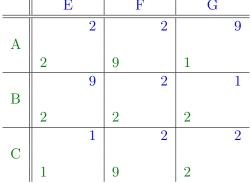
Consider the following 2-player bimatrix game.

The game is played repeatedly. Suppose the players choose the following sequence of strategies:

$$\binom{A}{E}, \binom{B}{F}, \binom{C}{G}, \binom{A}{E}, \binom{B}{F}, \binom{C}{G}, \dots$$

- a) Show that this sequence fulfills the no-regret property for both players.
- b) Prove or disprove: The average strategies of the two players in the given sequence converges to a mixed Nash equilibrium.
- c) Let  $\mathcal{V}$  be the probability distribution over states with  $\Pr_{s\sim\mathcal{V}}[s] = \frac{1}{3}$  for  $s \in \left\{\binom{A}{E}, \binom{B}{F}, \binom{C}{G}\right\}$  and 0 otherwise.

Prove or disprove:  $\mathcal{V}$  is a coarse-correlated equilibrium.





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(2 Points)

(2 + 2 + 2 Points)

#### Exercise 4.3.

Suppose  $\Gamma$  is a finite normal-form game, in which every player *i* has a strictly dominant strategy  $s_i^{DS} \in S_i$ , i.e.,

$$c_i(s_i, s_{-i}) > c_i(s_i^{DS}, s_{-i})$$
 for all  $s_i \in S_i \setminus \{s_i^{DS}\}, s_{-i} \in S_{-i}$ .

We call such a game *strictly dominant*. Let  $s^{DS}$  be the dominant-strategy equilibrium. Consider the distribution  $\mathcal{V}^{DS}$  with  $\Pr_{s\sim\mathcal{V}^{DS}}[s] = 1$  if  $s = s^{DS}$  and 0 otherwise.

Prove or disprove the following statements:

- a)  $\mathcal{V}^{DS}$  is a coarse-correlated equilibrium in every strictly dominant game  $\Gamma$ .
- b) There are strictly dominant games  $\Gamma$  with coarse-correlated equilibria  $\mathcal{V} \neq \mathcal{V}^{DS}$ .

#### Exercise 4.4.

(3 + 2 + 2 Points)

Consider a variant of the expert problem with N experts and two events  $\{Even, Odd\}$ . In every week, each expert predicts if the sum of points on this week's AGT sheet is even or odd. Denote with  $S_e^t$  the number of experts predicting *Even* for the *t*-th sheet.

Before publication of the t-th sheet, the algorithm POINTJORITY selects a prediction based on the experts' history and current guess. Therefore, expert i has a weight  $w_i^{t-1} \in [0,1]$  before sheet t is published, starting with  $w_i^0 = 1$  for all experts. Let  $W^{t-1} = \sum_{i=1}^N w_i^{t-1}$  be the sum of all weights before sheet t is published. The algorithm makes a majority decision based on weights, that is, it predicts *Even* for the t-th sheet if

$$\sum_{i \in S_e^t} w_i^{t-1} \ge W^{t-1}/2 \ ,$$

and Odd otherwise.

After publication of sheet t, the algorithm updates the weights of the experts: The weight of every expert i who gave a wrong prediction gets halved, i.e.,  $w_i^t = w_i^{t-1}/2$ .

a) Prove that if POINTJORITY makes a wrong prediction for sheet t, then

$$W^t \leq \frac{3}{4} \cdot W^{t-1}$$

b) Let f be the overall number of wrong predictions of POINTJORITY after T sheets have been published. Prove that

$$f \leq \log_{4/3}\left(\frac{W^0}{W^T}\right).$$

c) Let  $f_i$  be the overall number of wrong predictions of expert *i*. Use the statement in b) to prove that for each  $i \in N$ ,

$$f \leq \frac{1}{\ln(4/3)} \cdot (f_i + \ln N)$$

Assignments and further information concerning the course can be found at https://algo.cs.uni-frankfurt.de/lehre/agt/winter2324/agt2324.shtml

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