Algorithmic Game Theory

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Assignment 3

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Due: Nov 13, 2023, **23:55h**

Exercise 3.1. (2 + 3 + 3 Points)

Prove or disprove the following statements:

a) If Φ_1 and Φ_2 are two exact potential functions for the same game, then also Φ' is an exact potential for this game, where for every state s,

$$\Phi'(s) = \frac{\Phi_1(s) + \Phi_2(s)}{2} .$$

b) Every game with a pure Nash equilibrium is weakly acyclic.

c) Every game satisfying the finite improvement property has an ordinal potential.

Exercise 3.2. (3 + 2 Points)

Consider the following game called *fallout preparation*. There is an undirected graph $G = (V, E)$ given, where each vertex $v \in V$ represents a player. An edge $\{u, v\} \in E$ in the graph represents that two players u and v live in the neighborhood of each other. Every player can decide to build her own fallout shelter, or she can rely on the shelters of her neighbors. If a player decides to build her own shelter, she can pick between the two models `Survival` and `Comfort`. Building the model `Survival` has cost $C_{\texttt{surv}}$ while building the model `Comfort` has cost $C_{\texttt{comf}}$. If a player chooses to not build her own shelter instead, she obtains cost equal to the number of neighbors which also do not build any shelter (due to increased risk).

Formally, every player $v \in V$ has three strategies `SURV`, `COMF`, and `NONE`. The cost of player v in state s is given by

$$c_v(s) = \begin{cases} C_{\texttt{surv}} & \text{if } s_v = \texttt{SURV}, \\ C_{\texttt{comf}} & \text{if } s_v = \texttt{COMF}, \\ \left|\{\, \{v,w\} \in E \;:\; s_w = \texttt{NONE}\}\right| & \text{if } s_v = \texttt{NONE}. \end{cases}$$

a) Prove that the game is an exact potential game.

b) Prove that every sequence of best response improvement steps has length $\mathcal{O}(|V|^4)$.

Exercise 3.3. (4 Points)

Consider a congestion game $\Gamma = (\mathcal{N}, \mathcal{R}, (\sum_i)_{i\in\mathcal{N}}, (d_r)_{r\in\mathcal{R}})$. In the bottleneck variant of the game, each player minimizes the maximum delay on the selected resources instead of minimizing the sum of delays. Formally, the cost of player i in state S of the game is given by

$$c_i(S) \;=\; \max_{r\in S_i} d_r(n_r(S)),$$

where $n_r(S)$ is the number of players with $r \in S_i$.

Prove that every bottleneck congestion game with non-decreasing delay functions satisfies the finite improvement property.

Hint: Lexicographic potential.

Exercise 3.4. (3 + 2 + 2 Points)

Consider a matching game with the same number of men and women, i.e., $|\mathcal{X}| = |\mathcal{Y}|$. Note that there are matching games in which multiple stable matchings exist.

Let $x \in \mathcal{X}$ be a man in a matching game. A woman $y \in \mathcal{Y}$ is a *feasible partner* for x if there is a stable matching in which x and y are matched. Feasible partners for women are defined analogously.

a) Show that the Deferred Acceptance Algorithm with man proposal matches every man to his most preferred feasible partner.

b) Show that the Deferred Acceptance Algorithm with man proposal matches every woman to her least preferred feasible partner.

c) Design an algorithm that decides in polynomial time if there is a unique stable matching. Analyze the asymptotical running time and argue why your algorithm is correct.