Algorithmic Game Theory

Winter Term 2023/2024

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Assignment 2

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Due: Nov 6, 2023, **23:55h**

Exercise 2.1. (2 + 2 + 2 Points)

We generalize Sperner's Lemma to squares in the following way: We consider a square S that is subdivided into smaller squares by a grid of lines parallel to the edges of the original square. The vertices of the subdivision are the points of intersection of the lines. A *Sperner coloring* of S is a coloring of vertices that fulfills the following properties:

- The four outer corners of S are colored green, blue, orange, purple in clockwise order.
- Every vertex on the boundary (i.e. the outer side) of S is colored with one of the two colors of the endpoints of the corresponding outer line.
- Vertices in the interior of S are colored arbitrarily in one of the four colors.

An edge between a green and a blue vertex is called a *door*. Doors on the boundary of S are called *entrances*. Show the following properties for a Sperner coloring of a square S:

a) There is an odd number of entrances.

b) There is at least one small square with at least three different colors.

c) There is an odd number of small squares, with at least three different colors and exactly one door.

Exercise 2.2. (2 + 3 Points)

Suppose $(a_{\mathtt{I}}, a_{\mathtt{II}})$ and $(b_{\mathtt{I}}, b_{\mathtt{II}})$ are two mixed Nash equilibria in a 2-player zero-sum game with players I, II and their respective strategy sets $S_{\mathtt{I}}$, $S_{\mathtt{II}}$.

a) Are $(a_{\mathtt{I}}, b_{\mathtt{II}})$ and $(b_{\mathtt{I}}, a_{\mathtt{II}})$ mixed Nash equilibria? Prove your answer.

b) Show that the state $(x_{\mathtt{I}}, x_{\mathtt{II}})$ with

$$x_{\mathtt{I},j} = \frac{1}{2} \cdot (a_{\mathtt{I},j} + b_{\mathtt{I},j}) \qquad \text{for all } j \in \{1, \dots, |S_{\mathtt{I}}|\},$$
$$x_{\mathtt{II},j} = \frac{1}{2} \cdot (a_{\mathtt{II},j} + b_{\mathtt{II},j}) \qquad \text{for all } j \in \{1, \dots, |S_{\mathtt{II}}|\}$$

is a mixed Nash equilibrium.

Exercise 2.3. (2 + 3 + 2 Points)

Consider the following game:

	S	C
S	a, a	d, c
C	c, d	b, b

a) For values $a = 2, b = 4, c = 1$ and $d = 5$, the game equals the prisoner's dilemma. Construct an exact potential function for this game.

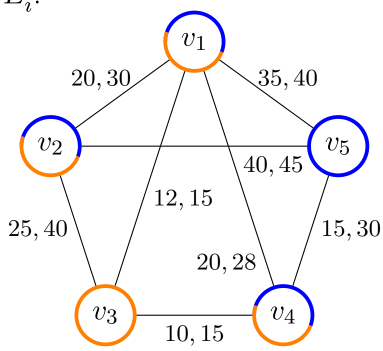
b) Prove that the game is an exact potential game, for every $a, b, c, d \in \mathbb{R}$.

c) Construct a 2×2-game (2 players, 2 strategies) with the following properties:

- The game has a pure Nash equilibrium.
- The game does not have an exact potential function.

Exercise 2.4. (2 + 3 Points)

A *connection* game is a congestion game with n agents and an undirected graph $G = (V, E)$. Every agent i is associated with a subset of vertices $V_i \subseteq V$. The set of strategies Σ_i consists of all connected, acyclic subgraphs G'_i with $V'_i = V_i$ and $E'_i \subseteq (E \cap (V_i \times V_i))$, for every player i. Every edge e is assigned a delay function $d_e(n_e) : \{1, \ldots, n\} \to \mathbb{Z}$, where n_e is the number of agents i selecting a subgraph G'_i with $e \in E'_i$.



a) Consider the above instance of a connection game with two players. The vertices in V_1 are indicated in orange, while the vertices in V_2 are marked in blue. Let the initial strategie of player 1 be given by the subgraph G'_1 with edges $E'_1 = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}\}$. Player 2 chooses subgraph G'_2 with $E'_2 = \{\{v_1, v_5\}, \{v_2, v_5\}, \{v_4, v_5\}\}$ as his strategy.
Perform best-response improvement steps until a pure Nash equilibrium is reached. Player 1 should deviate first.

b) Prove: Every sequence of best-response improvement steps in a connection game converges in $O(n^2 \cdot |E| \cdot |V|)$ many steps.
Hint: You can use the following property without proving it.
Let G'_i be the strategy of agent i in state S, and let G''_i be a best response of i for S_{-i}. Then, there exists a transforming sequence from G'_i to G''_i, where in every step, one edge $e' \in (E'_i \setminus E''_i)$ is exchanged by an edge $e'' \in (E''_i \setminus E'_i)$. For each step, the resulting graph is a feasible strategy for agent i. In particular, the delay is (weakly) reduced in every step.

Assignments and further information concerning the course can be found at
`https://algo.cs.uni-frankfurt.de/lehre/agt/winter2324/agt2324.shtml`

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