Algorithmic Game Theory

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Assignment 10

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Due: Jan 24, 2023, **10:00h**

Exercise 10.1. (2 + 2 + 2 Points)

Prove the following statements:

a) The plurality rule does not always fulfill the Condorcet-winner criterion.

b) A social choice function is incentive compatible if and only if it is monotone.

c) For an odd number of voters and single-peaked preferences, the median mechanism (i.e. the k-th order mechanism with $k = \lfloor (n+1)/2 \rfloor$) fulfills the Condorcet-winner criterion.

Exercise 10.2. (1 + 2 Points)

We consider a social choice function that iteratively eliminates candidates to select a winner. In each step, an arbitrary pair of (remaining) candidates is compared using the majority rule and the candidate that is not the winner is eliminated. The process stops when there is no pair of candidates remaining such that one candidate can be eliminated. If only a single candidate remains, she is then selected as the winning candidate. Otherwise, there is not a winner. Note that this social choice function (i.e. the order of comparisions) can be represented by a tournament tree.

a) Prove: There is always a winner (for every possible tournament tree) if and only if n is odd.

b) Consider the following instance with candidate set $A = \{a, b, c, d\}$:

$$a \succ_1 c \succ_1 d \succ_1 b,$$
$$c \succ_2 b \succ_2 a \succ_2 d.$$

Construct the preference order $\succ_3$ of a third voter that fulfills both of the following requirements:

- For each candidate $x \in A$, there is a tournament tree where x wins.
- The Borda Count of each pair of candidates differs by at most 1.

Prove the correctness of your construction.

Exercise 10.3. (2 + 2 + 2 + 2 + 2 Points)

Consider a beach that can be represented by the interval $[0, 1]$. There are n people visiting the beach and visitor i has a most favorite spot $s_i \in [0, 1]$. We would like to place ice-cream sellers at the beach. We ask every visitor for the most preferred position $b_i \in [0, 1]$ and each visitor i has an incentive that an ice-cream seller gets placed as close as possible to s_i. Let $b = (b_1, \ldots, b_n)$, and $\mathcal{N} = \{1, \ldots, n\}$.

First, assume that we only place a single ice-cream seller at position $p_1 \in [0, 1]$.

a) Let $d^\Sigma(p_1, b)$ be the total distance of all reported positions to the ice-cream seller at p_1, i.e.,

$$d^\Sigma(p_1, b) = \sum_{i=1}^{n} |b_i - p_1| \ .$$

Prove or disprove: There is an incentive-compatible mechanism without money such that $d^\Sigma(p_1, b)$ is minimized.

b) Consider the maximum distance of any visitor to p_1, i.e.,

$$d^{\max}(p_1, b) = \max_{i \in \mathcal{N}} |b_i - p_1| \ .$$

Prove or disprove: There is an incentive-compatible mechanism without money such that $d^{\max}(p_1, b)$ is minimized.

For the following tasks, assume we place two ice-cream sellers at $p_1, p_2 \in [0, 1]$.

c) Consider again the maximum distance of any visitor to the next ice-cream seller. Let

$$d^{\max}(p_1, p_2, b) = \max_{i \in \mathcal{N}} \{\min(|b_i - p_1|, |b_i - p_2|)\} \ .$$

Prove or disprove: There is an incentive-compatible mechanism without money such that $d^{\max}(p_1, p_2, b)$ is minimized.

d) Consider the following max-min-mechanism: Choose $p_1 = \min_{i \in \mathcal{N}} b_i$ and $p_2 = \max_{i \in \mathcal{N}} b_i$. Prove or disprove: This mechanism is incentive compatible.

e) Prove that the max-min-mechanism is a 2-approximation for the maximum distance, i.e.,

$$d^{\max}(p_1, p_2, b) \quad \leq \quad 2 \cdot \min_{q_1, q_2 \in [0,1]} d^{\max}(q_1, q_2, b) \ .$$