Algorithmic Game Theory

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Assignment 4

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In many real-world scenarios that can be modeled as congestion games, there are different types of players to consider. For example, roads might be used by different types of vehicles (bikes, cars, busses, ...) with different properties. Even vehicles of the same type might differ in their size and their impact on the traffic situation.
On this exercise sheet, we study an extension for congestion games that models this player heterogeneity by adding player weights.

Definition. A *weighted congestion game* is a tuple $(\mathcal{N}, (w_i)_{i\in\mathcal{N}}, \mathcal{R}, (\Sigma_i)_{i\in\mathcal{N}}, (d_r)_{r\in\mathcal{R}})$ where

- $\mathcal{N} = \{1, \dots, n\}$ is the set of players,
- $w_i \in \mathbb{R}_{>0}$ is the weight of player $i \in \mathcal{N}$,
- $\mathcal{R}$ is the set of resources,
- $\Sigma_i \subseteq 2^{\mathcal{R}}$ is the strategy space of player $i \in \mathcal{N}$,
- $d_r : \mathbb{R} \to \mathbb{R}$ is the delay function of resource $r \in \mathcal{R}$,
- $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ denotes the state space.

For every state $S = (S_1, \dots, S_n) \in \Sigma$, the cost function for any player $i \in \mathcal{N}$ is given by

$$c_i(S) := \sum_{r\in S_i} w_i \cdot d_r\left(m_r(S)\right) \quad \text{with} \quad m_r(S) := \sum_{\substack{j\in\mathcal{N}:\\ r\in S_j}} w_j.$$

Exercise 4.1. (1 + 4 Points)

a) Prove that every congestion game corresponds to a weighted congestion game.

b) Construct a weighted congestion game with two players of weights 1 and 2 (respectively) and resources $\mathcal{R} = \{a, b, c\}$ that does not have a pure Nash equilibrium.

   Prove the correctness of your construction.

   Hint: Assume $\Sigma_1 = \{\{a\}, \{b, c\}\}$ and $\Sigma_2 = \{\{b\}, \{a, c\}\}$ and choose appropriate delay functions.

Exercise 4.2. (3 + 3 Points)

We want to show that the function $\Phi : \Sigma \to \mathbb{R}$ with

$$\Phi(S) = \sum_{r \in \mathcal{R}} \sum_{\substack{i \in \mathcal{N}: \\ r \in S_i}} w_i \cdot d_r \left(\sum_{\substack{k \in \{1,\dots,i\}: \\ r \in S_k}} w_k \right)$$

is an exact potential function for weighted congestion games that are restricted to **affine delay functions**, i.e. for each $r \in \mathcal{R}$, there are $\alpha_r, \beta_r \in \mathbb{R}$ such that $d_r(x) = \alpha_r \cdot x + \beta_r$.

Prove the following statements for weighted congestion games:

a) For any two strategies $S_n, S'_n \in \Sigma_n$ **of the last player** n, it holds:

$$\Phi(S_n, S_{-n}) - \Phi(S'_n, S_{-n}) = c_n(S_n, S_{-n}) - c_n(S'_n, S_{-n})$$

b) If there is a restriction to affine delay functions, then $\Phi(S)$ is independent of the player order.

Hint: Show that $\Phi(S)$ can be expressed in a different way where the player order is not relevant.

Be prepared to argue in the exercise session: Why is it sufficient to show these two steps?

Exercise 4.3. (4 Points)

A weighted congestion game is called *singleton* if for all $i \in \mathcal{N}$, it holds $|S_i| = 1$ for all $S_i \in \Sigma_i$.

A *load balancing game* is a special case of a singleton weighted congestion game where each delay function d_r is the identity function (i.e. $d_r(x) = x$ for all $x \in \mathbb{R}$).

Prove that load balancing games are ordinal potential games.

Hint: Construct an ordinal potential function with lexicographical decrease and prove its correctness.
