Strategic Payments in Financial Networks

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Systemic Risk in Financial Networks

Financial crisis in 2008:

- Cascading defaults and bankruptcies

Design and analyze a model with

- financial entities (banks, investment funds, etc.)
- monetary liabilities and dependencies

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Main Goals:

- Analyze debts as major source of risk in financial systems.
- Understand effects and design suitable measures for regulation of financial markets.

Money Flow Games

Eisenberg-Noe Model [Eisenberg, Noe, 2001]

- Set V of n financial institutions or firms
- Set E of directed edges, edge $e \in E$ has value $c_e > 0$
- $e = (u, v)$ represents a debt of c_e that u owes to v
- Each institution has liquid assets of value $a_v^l \geq 0$.
- Call $l(v) = \sum_{e \in E^+(v)} c_e$ the liabilities of v.

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Strategic Choices

- Every firm chooses a flow function $f_e : \mathbb{N} \to \mathbb{N}$ for each outgoing edge.
- This specifies how each firm's assets are distributed.
- f_e fulfills
  - $f_e(y) \leq f_e(z)$ for all $e \in E^+(v)$ and $0 \leq y \leq z$. (non-decreasing)
  - $0 \leq f_e(y) \leq c_e$ for all $e \in E^+(v)$ and $y \in \mathbb{N}$. (capacity constraint)
  - $\sum_{e \in E^+(v)} f_e(y) = \min\{y, l(v)\}$. (no-fraud constraint)

Clearing States

[Eisenberg, Noe, 2001]

Given the strategy choices f_e of the players, a clearing state $\mathbf{a} = (a_v)_{v \in V}$ is a vector of assets that obeys the strategy choices of the firms, i.e.

$$a_v = a_v^l + \sum_{e=(u,v) \in E^-(v)} f_e(a_u) \ .$$

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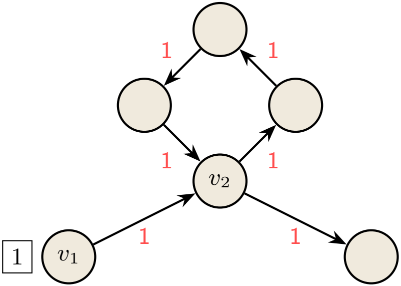

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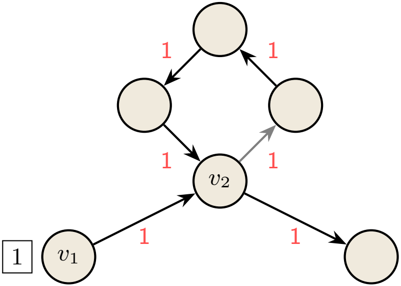

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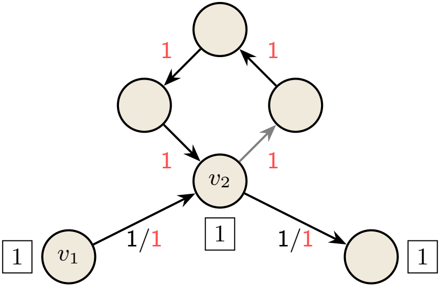

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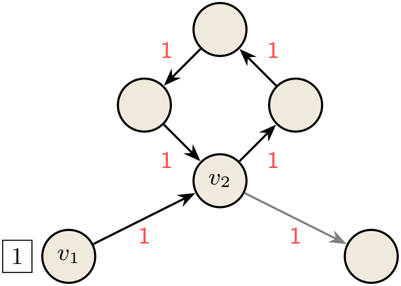

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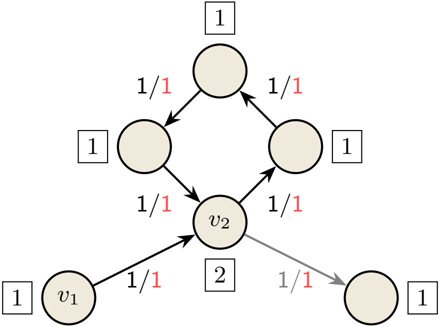

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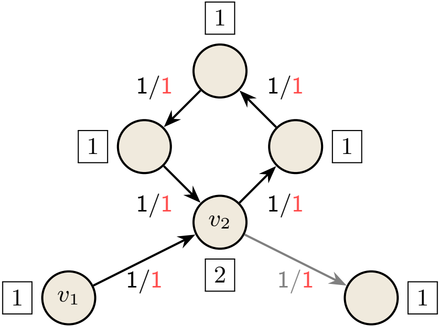

The firms strategically seek to maximize their assets, i.e. they try to clear as much of their debts as possible.

Related Work

Financial Networks:

- Network model
  [Eisenberg, Noe, 2001]
- Computational complexity of finding clearing states with credit default swaps
  [Seuken, Schuldenzucker, Battiston, 2017]
- Estimating the number of defaults
  [Hemenway, Khanna, 2016]

Flow Games

- Strategic max-flow games
  [Kupferman et al, 2017, 2018]
- Stable flows
  [Fleiner, 2014; Cseh, Matuschke, 2019]

Strategic Payments

Main Objectives

- Existence and uniqueness of clearing states.
- Analyze how clearing states correspond to each other.

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- Existence and uniqueness of clearing states.
- Analyze how clearing states correspond to each other.

For a uniquely defined clearing state:

- Do Nash equilibria always exist?
- What is the computational complexity of finding a Nash equilibrium?
- Analyze the inefficiency of Nash equilibria.

Edge- & Coin-Ranking Games

Edge-Ranking: Ranking π_v over outgoing edges $E^+(v)$. Debt is payed according to this ranking.

Coin-Ranking: Arbitrary strategies. Integrality of c_e and a_v^l: Ranking π_v over integral flow units (i.e., coins) spent on the outgoing edges $E^+(v)$

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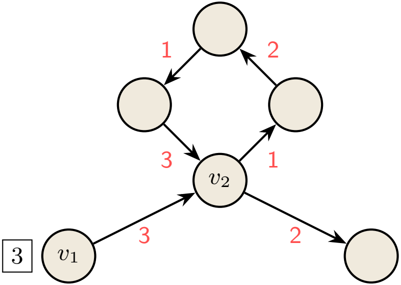

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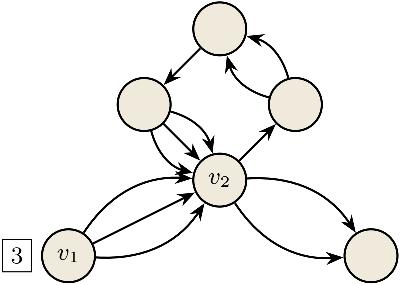

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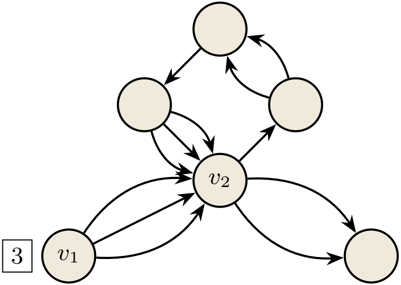

Coin-Ranking = Edge-Ranking in a graph with uniform edge weights.
(Transformation just for intuition not necessarily poly-time...)

Existence of Clearing States

Theorem

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Proof.

- Top preferences yield unique set of paths and cycles.
- Choose a node v at which the clearing state condition is not fulfilled. Push flow until some edge goes tight.
- Edge (u, w) goes tight $\rightarrow$ u switches to next-higher ranked edge in π_u.

□

Related to Top-Trading-Cycles algorithm [Shapley, Scarf, 1974]

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Observation

Clearing states are not unique.

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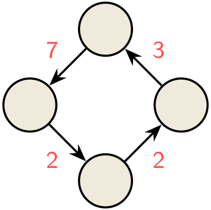

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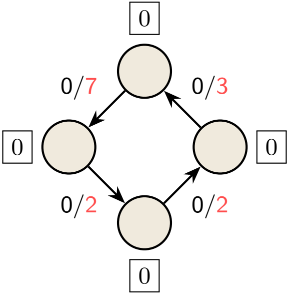

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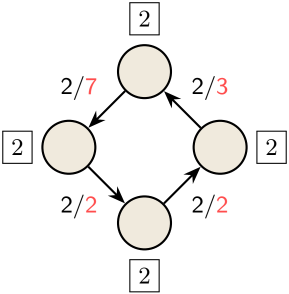

Structural Properties of Clearing States

Let $\mathcal{A}$ be the set of feasible clearing states for a strategy profile $\mathbf{f}$.

Theorem

For every strategy profile in a money-flow game the pair $(\mathcal{A}, \leq)$ *forms a complete lattice.*

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Let $(\mathcal{A}, \leq)$ *be any complete lattice. Suppose* $g : \mathcal{A} \to \mathcal{A}$ *is order-preserving, i.e., for all* $x, y \in \mathcal{A}$, $x \leq y$ *implies* $g(x) \leq g(y)$*. Then the set of all fixed points of* g *is a complete lattice with respect to* $\leq$*.*

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Proof of Theorem.

- Let $A = \{\mathbf{a} \mid 0 \leq a_v \leq a_v^l + \sum_{e \in E^-(v)} c_e \forall v \in V\}$. $(A, \leq)$ forms a complete lattice.
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- Apply Knaster-Tarski.

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For the rest of the talk, always choose the unique maximum clearing state!

Results for Coin-Ranking Games

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Every money flow game has a strong equilibrium. The strong PoS is 1. This equilibrium can be calculated in polynomial time.

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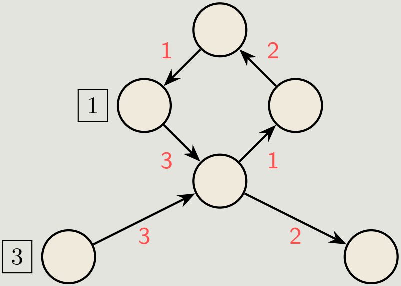

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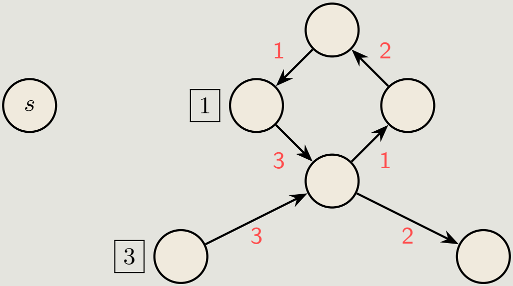

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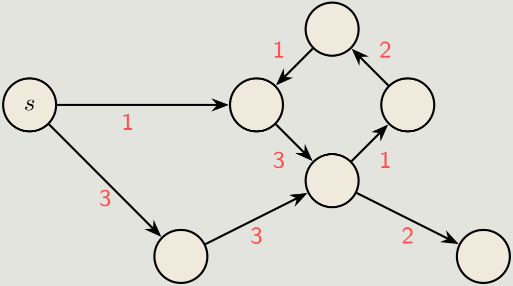

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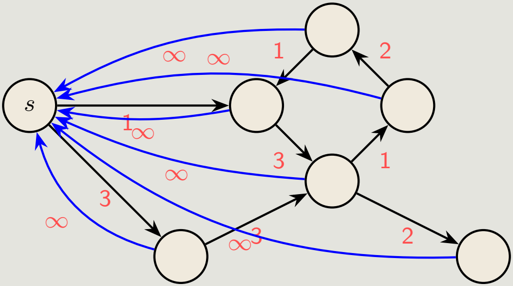

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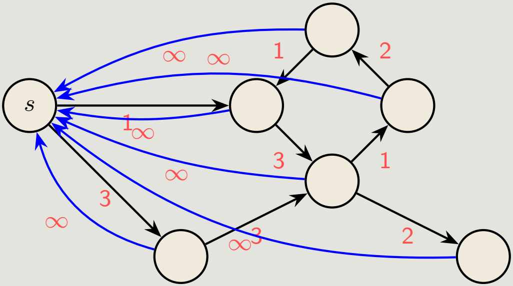

Observe: Optimal circulation maximizes assets in the original graph. This can be calculated in polynomial time. [Tardos, 1985]

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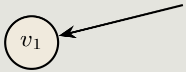

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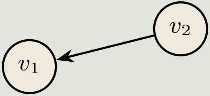

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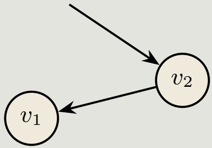

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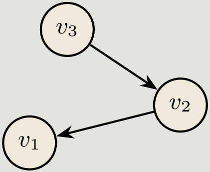

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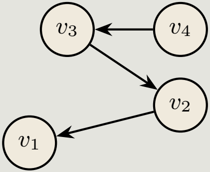

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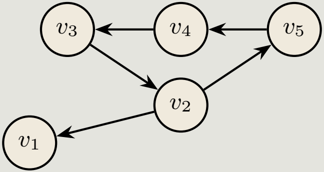

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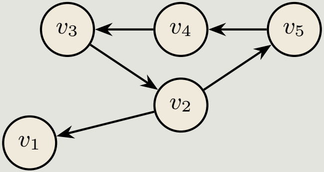

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Theorem

The PoA is unbounded.

Results on Coin-Ranking Games:

- Strong equilibria exist. Computation in polynomial time.
- Strong PoS $= 1$.
- PoA unbounded.

Edge- & Coin-Ranking Games

Results on Coin-Ranking Games:

- Strong equilibria exist. Computation in polynomial time.
- Strong PoS $= 1$.
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Results on Edge-Ranking Games:

- Nash equilibria might be absent.
- Strong PoS unbounded.
- Deciding whether a strong / Nash equilibrium exists is NP-hard.
- Computing a strategy profile with maximum total revenue.

The restriction to edge-ranking games is harmful!

Bankruptcy Settlement

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